5 Matematička indukcija

Zadatak 5.1 *Dokazati da za svaki prirodan broj n vrijedi*

$$1+3+5+\cdots+(2n-1)=n^2.$$

Rješenje:

1) Provjerimo da li jednakost vrijedi za $n=1$:
$\mathcal{L}\equiv 2\cdot 1-1=2-1=1$
$\mathcal{D}\equiv 1^2=1$
Dakle, jednakost vrijedi za $n=1$.

2) Pretpostavimo da tvrdnja vrijedi za $n=k$, tj. neka vrijedi

$$1+3+5+\cdots+(2k-1)=k^2.$$

3) Dokažimo da tvrdnja vrijedi i za $n=k+1$, tj. da vrijedi

$$1+3+5+\cdots+(2(k+1)-1)=(k+1)^2.$$

Dokaz

$$\begin{aligned}1+3+5+\cdots+(2k-1)+(2(k+1)-1) &= k^2+(2(k+1)-1)=\\ &= k^2+2k+1=(k+1)^2.\end{aligned}$$

Dakle, jednakost vrijedi i za $n=k+1$.

4) Jednakost vrijedi za svaki prirodan broj n.

Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n}(2k-1)=n^2.$$

Zadatak 5.2 *Dokazati da za svaki prirodan broj n vrijedi*

$$1^3+2^3+3^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2.$$

Rješenje:

1) Provjerimo da li jednakost vrijedi za $n=1$:
$\mathcal{L}\equiv 1^3=1$
$\mathcal{D}\equiv \left(\frac{1(1+1)}{2}\right)^2=1^2=1$
Dakle, jednakost vrijedi za $n=1$.

2) Pretpostavimo da tvrdnja vrijedi za $n=k$, tj. neka vrijedi

$$1^3+2^3+3^3+\cdots+k^3=\left(\frac{k(k+1)}{2}\right)^2.$$

3) Dokažimo da tvrdnja vrijedi i za $n=k+1$, tj. da vrijedi

$$1^3+2^3+3^3+\cdots+(k+1)^3=\left(\frac{(k+1)((k+1)+1)}{2}\right)^2$$

Dokaz:

$$\begin{aligned}1^3+2^3+3^3+\cdots+k^3+(k+1)^3 &= \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3=\\ &= \frac{k^2(k+1)^2}{4}+(k+1)^3=\\ &= \frac{k^2(k+1)^2+4(k+1)^3}{4}=\\ &= \frac{(k+1)^2(k^2+4(k+1))}{4}=\\ &= \frac{(k+1)^2(k^2+4k+4)}{4}=\\ &= \frac{(k+1)^2(k+2)^2}{4}=\\ &= \left(\frac{(k+1)(k+2)}{2}\right)^2=\\ &= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2.\end{aligned}$$

Dakle, jednakost vrijedi i za $n=k+1$.

4) Jednakost vrijedi za svaki prirodan broj n.

Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n}k^3=\left(\frac{n(n+1)}{2}\right)^2.$$

Zadatak 5.3 *Dokazati da za svaki prirodan broj n vrijedi*

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)\cdot(2n+1)}=\frac{n}{2n+1}.$$

Rješenje:

1) Provjerimo da li jednakost vrijedi za $n = 1$:

$\mathcal{L} \equiv \frac{1}{1\cdot 3} = \frac{1}{3}$

$\mathcal{D} \equiv \frac{1}{2\cdot 1+1} = \frac{1}{3}$

Dakle, jednakost vrijedi za $n = 1$.

2) Pretpostavimo da tvrdnja vrijedi za $n = k$, tj. neka vrijedi

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2k-1)\cdot(2k+1)} = \frac{k}{2k+1}.$$

3) Dokažimo da tvrdnja vrijedi i za $n = k+1$, tj. da vrijedi

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2(k+1)-1)\cdot(2(k+1)+1)} = \frac{(k+1)}{2(k+1)+1}.$$

Dokaz:

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2k-1)\cdot(2k+1)} + \frac{1}{(2(k+1)-1)\cdot(2(k+1)+1)} =$$

$$\begin{aligned}
&= \frac{k}{2k+1} + \frac{1}{(2(k+1)-1)\cdot(2(k+1)+1)} = \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)\cdot(2k+3)} = \frac{k(2k+3)+1}{(2k+1)\cdot(2k+3)} = \\
&= \frac{2k^2+3k+1}{(2k+1)\cdot(2k+3)} = \frac{(2k+1)\cdot(k+1)}{(2k+1)\cdot(2k+3)} = \frac{(k+1)}{(2k+3)} = \frac{(k+1)}{2(k+1)+1}.
\end{aligned}$$

Dakle, jednakost vrijedi i za $n = k+1$.

Rješenja kvadratne jednačine $2k^2 + 3k + 1 = 0$ su brojevi $k_1 = -\frac{1}{2}$ i $k_2 = -1$, pa je $2k^2 + 3k + 1 = 2\cdot\left(k+\frac{1}{2}\right)\cdot(k+1) = (2k+1)\cdot(k+1)$.

4) Jednakost vrijedi za svaki prirodan broj n.

Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n} \frac{1}{(2k-1)\cdot(2k+1)} = \frac{n}{2n+1}.$$

Zadatak 5.4 *Dokazati da za svaki prirodan broj n vrijedi*

$$\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \cdots + \frac{n^2}{(2n-1)\cdot(2n+1)} = \frac{n(n+1)}{2(2n+1)}.$$

Rješenje:

1) Provjerimo da li jednakost vrijedi za $n = 1$:

$\mathcal{L} \equiv \frac{1^2}{1\cdot 3} = \frac{1}{3}$

$\mathcal{D} \equiv \frac{1\cdot(1+1)}{2\cdot(2\cdot 1+1)} = \frac{1}{3}$

Dakle, jednakost vrijedi za $n = 1$.

2) Pretpostavimo da tvrdnja vrijedi za $n = k$, tj. neka vrijedi

$$\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \cdots + \frac{k^2}{(2k-1)\cdot(2k+1)} = \frac{k(k+1)}{2(2k+1)}.$$

3) Dokažimo da tvrdnja vrijedi i za $n = k+1$, tj. da vrijedi

$$\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \cdots + \frac{(k+1)^2}{(2(k+1)-1)\cdot(2(k+1)+1)} = \frac{(k+1)((k+1)+1)}{2(2(k+1)+1)}.$$

Dokaz:

$$\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \cdots + \frac{k^2}{(2k-1)\cdot(2k+1)} + \frac{(k+1)^2}{(2(k+1)-1)\cdot(2(k+1)+1)} =$$

$$\begin{aligned}
&= \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2(k+1)-1)(2(k+1)+1)} = \\
&= \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)} = \\
&= \frac{k(k+1)(2k+3)+2(k+1)^2}{2(2k+1)(2k+3)} = \frac{(k+1)(k(2k+3)+2(k+1))}{2(2k+1)(2k+3)} = \\
&= \frac{(k+1)(2k^2+3k+2k+2)}{2(2k+1)(2k+3)} = \frac{(k+1)(2k^2+5k+2)}{2(2k+1)(2k+3)} = \\
&= \frac{(k+1)(k+2)(2k+1)}{2(2k+1)(2k+3)} = \frac{(k+1)(k+2)}{2(2k+3)} = \frac{(k+1)((k+1)+1)}{2(2(k+1)+1)}.
\end{aligned}$$

Dakle, jednakost vrijedi i za $n = k+1$.

Rješenja kvadratne jednačine $2k^2 + 5k + 1 = 0$ su brojevi $k_1 = -\frac{1}{2}$ i $k_2 = -2$, pa je $2k^2 + 5k + 1 = 2\cdot\left(k+\frac{1}{2}\right)\cdot(k+2) = (2k+1)\cdot(k+2)$.

4) Jednakost vrijedi za svaki prirodan broj n.

Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n} \frac{k^2}{(2k-1)\cdot(2k+1)} = \frac{n(n+1)}{2(2n+1)}.$$

Zadatak 5.5 *Dokazati da za svaki prirodan broj n vrijedi*

$$\frac{3}{4}+\frac{5}{36}+\cdots+\frac{2n+1}{n^2\left(n+1\right)^2}=1-\frac{1}{\left(n+1\right)^2}.$$

Rješenje:

1) Provjerimo da li jednakost vrijedi za $n=1$:

$\mathcal{L}\equiv\frac{2\cdot1+1}{1^2(1+1)^2}=\frac{3}{4}$

$\mathcal{D}\equiv1-\frac{1}{(1+1)^2}=1-\frac{1}{4}=\frac{3}{4}.$

Dakle, jednakost vrijedi za $n=1$.

2) Pretpostavimo da tvrdnja vrijedi za $n=k$, tj. neka vrijedi

$$\frac{3}{4}+\frac{5}{36}+\cdots+\frac{2k+1}{k^2\left(k+1\right)^2}=1-\frac{1}{\left(k+1\right)^2}.$$

3) Dokažimo da tvrdnja vrijedi i za $n=k+1$, tj. da vrijedi

$$\frac{3}{4}+\frac{5}{36}+\cdots+\frac{2\left(k+1\right)+1}{\left(k+1\right)^2\left(\left(k+1\right)+1\right)^2}=1-\frac{1}{\left(\left(k+1\right)+1\right)^2}.$$

Dokaz:

$$\frac{3}{4}+\frac{5}{36}+\cdots+\frac{2k+1}{k^2\left(k+1\right)^2}+\frac{2\left(k+1\right)+1}{\left(k+1\right)^2\left(\left(k+1\right)+1\right)^2}=$$

$$\begin{aligned}
&= 1-\frac{1}{\left(k+1\right)^2}+\frac{2\left(k+1\right)+1}{\left(k+1\right)^2\left(\left(k+1\right)+1\right)^2}=\\
&= 1-\frac{1}{\left(k+1\right)^2}+\frac{2k+3}{\left(k+1\right)^2\left(k+2\right)^2}=\\
&= 1-\frac{\left(k+2\right)^2-\left(2k+3\right)}{\left(k+1\right)^2\left(k+2\right)^2}=1-\frac{k^2+4k+4-2k-3}{\left(k+1\right)^2\left(k+2\right)^2}=\\
&= 1-\frac{k^2+2k+1}{\left(k+1\right)^2\left(k+2\right)^2}=1-\frac{\left(k+1\right)^2}{\left(k+1\right)^2\left(k+2\right)^2}=\\
&= 1-\frac{1}{\left(k+2\right)^2}=1-\frac{1}{\left(\left(k+1\right)+1\right)^2}.
\end{aligned}$$

Dakle, jednakost vrijedi i za $n=k+1$.

4) Jednakost vrijedi za svaki prirodan broj n.

Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n}\frac{2k+1}{k^2\left(k+1\right)^2}=1-\frac{1}{\left(n+1\right)^2}.$$

Zadatak 5.6 *Dokazati da za svaki prirodan broj n vrijedi*

$$\frac{5}{1\cdot2}+\frac{12}{2\cdot3}+\cdots+\frac{2n^2+2n+1}{n\cdot\left(n+1\right)}=\frac{n\left(2n+3\right)}{n+1}.$$

Rješenje:

1) Provjerimo da li jednakost vrijedi za $n=1$:

$\mathcal{L}\equiv\frac{5}{1\cdot2}=\frac{5}{2}$

$\mathcal{D}\equiv\frac{1(2\cdot1+3)}{1+1}=\frac{5}{2}$

Dakle, jednakost vrijedi za $n=1$.

2) Pretpostavimo da tvrdnja vrijedi za $n=k$, tj. neka vrijedi

$$\frac{5}{1\cdot2}+\frac{12}{2\cdot3}+\cdots+\frac{2k^2+2k+1}{k\cdot\left(k+1\right)}=\frac{k\left(2k+3\right)}{k+1}.$$

3) Dokažimo da tvrdnja vrijedi i za $n=k+1$, tj. da vrijedi

$$\frac{5}{1\cdot2}+\frac{12}{2\cdot3}+\cdots+\frac{2\left(k+1\right)^2+2\left(k+1\right)+1}{\left(k+1\right)\cdot\left(\left(k+1\right)+1\right)}=\frac{\left(k+1\right)\left(2\left(k+1\right)+3\right)}{\left(k+1\right)+1}$$

Dokaz:

$$\frac{5}{1\cdot2}+\frac{12}{2\cdot3}+\cdots+\frac{2k^2+2k+1}{k\cdot\left(k+1\right)}+\frac{2\left(k+1\right)^2+2\left(k+1\right)+1}{\left(k+1\right)\cdot\left(\left(k+1\right)+1\right)}=$$

$$
\begin{aligned}
&= \frac{k\,(2k+3)}{k+1} + \frac{2\,(k+1)^2 + 2\,(k+1) + 1}{(k+1)\cdot((k+1)+1)} = \\
&= \frac{k\,(2k+3)}{k+1} + \frac{2\,(k^2+2k+1) + 2\,(k+1) + 1}{(k+1)\cdot(k+2)} = \\
&= \frac{2k^2+3k}{k+1} + \frac{2k^2+4k+2+2k+2+1}{(k+1)\cdot(k+2)} = \\
&= \frac{2k^2+3k}{k+1} + \frac{2k^2+6k+5}{(k+1)\cdot(k+2)} = \\
&= \frac{(2k^2+3k)\,(k+2) + 2k^2+6k+5}{(k+1)\cdot(k+2)} = \\
&= \frac{2k^3+4k^2+3k^2+6k+2k^2+6k+5}{(k+1)\cdot(k+2)} = \\
&= \frac{2k^3+9k^2+12k+5}{(k+1)\cdot(k+2)} = \frac{2k^3+2k^2+7k^2+7k+5k+5}{(k+1)\cdot(k+2)} = \\
&= \frac{2k^2\,(k+1)+7k\,(k+1)+5\,(k+1)}{(k+1)\cdot(k+2)} = \\
&= \frac{(2k^2+7k+5)\cdot(k+1)}{(k+1)\cdot(k+2)} = \frac{2k^2+7k+5}{k+2} = \\
&= \frac{2k^2+2k+5k+5}{k+2} = \frac{(k+1)\,(2k+5)}{k+2} = \\
&= \frac{(k+1)\,(2\,(k+1)+3)}{(k+1)+1}
\end{aligned}
$$

Dakle, jednakost vrijedi i za $n = k+1$.
4) Jednakost vrijedi za svaki prirodan broj n.
Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n} \frac{2k^2+2k+1}{k\cdot(k+1)} = \frac{n\,(2n+3)}{n+1}.$$

Zadatak 5.7 *Dokazati da za svaki prirodan broj n vrijedi*

$$17 \mid 5^{n+3} + 11^{3n+1}.$$

Rješenje:
1) Provjerimo da li jednakost vrijedi za $n = 1$:

$$5^{1+3} + 11^{3\cdot1+1} = 5^4 + 11^4 = 625 + 14641 = 15266 = 17\cdot 898$$

Dakle, jednakost vrijedi za $n = 1$.
2) Pretpostavimo da tvrdnja vrijedi za $n = k$, tj. neka vrijedi

$$17 \mid 5^{k+3} + 11^{3k+1} \implies 5^{k+3} + 11^{3k+1} = 17A.$$

3) Dokažimo da tvrdnja vrijedi i za $n = k+1$, tj. da vrijedi

$$17 \mid 5^{(k+1)+3} + 11^{3(k+1)+1}$$

Dokaz:

$$
\begin{aligned}
5^{(k+1)+3} + 11^{3(k+1)+1} &= 5^{k+4} + 11^{3k+4} = 5^{k+3}\cdot 5 + 11^{3k+1}\cdot 11^3 = \\
&= 5^{k+3}\cdot 5 + 11^{3k+1}\cdot 1331 = \\
&= 5^{k+3}\cdot 5 + 11^{3k+1}\cdot (1326+5) = \\
&= 5^{k+3}\cdot 5 + 11^{3k+1}\cdot 5 + 11^{3k+1}\cdot 1326 = \\
&= 5\cdot\left(5^{k+3} + 11^{3k+1}\right) + 11^{3k+1}\cdot 1326 = \\
&= 5\cdot 17A + 11^{3k+1}\cdot 17\cdot 78 = \\
&= 17\left(5\cdot A + 11^{3k+1}\cdot 78\right) = 17B.
\end{aligned}
$$

gdje je $B = 5\cdot A + 11^{3k+1}\cdot 78$.
Dakle, jednakost vrijedi i za $n = k+1$.
4) Jednakost vrijedi za svaki prirodan broj n.

Zadatak 5.8 *Dokazati da za svaki prirodan broj n vrijedi*

$$19 \mid 7\cdot 5^{2n} + 12\cdot 6^n.$$

Rješenje:
1) Provjerimo da li jednakost vrijedi za $n = 1$:

$$7\cdot 5^{2\cdot 1} + 12\cdot 6^1 = 7\cdot 25 + 12\cdot 6 = 175 + 72 = 247 = 19\cdot 13.$$

Dakle, jednakost vrijedi za $n = 1$.
2) Pretpostavimo da tvrdnja vrijedi za $n = k$, tj. neka vrijedi

$$19 \mid 7\cdot 5^{2n} + 12\cdot 6^n \implies 7\cdot 5^{2n} + 12\cdot 6^n = 19A.$$

3) Dokažimo da tvrdnja vrijedi i za $n = k+1$, tj. da vrijedi

$$19 \mid 7\cdot 5^{2(n+1)} + 12\cdot 6^{(n+1)}$$

Dokaz:

$$\begin{aligned}
7\cdot 5^{2(n+1)}+12\cdot 6^{(n+1)} &= 7\cdot 5^{2n+2}+12\cdot 6^{n+1}=7\cdot 5^{2n}\cdot 25+12\cdot 6^{n}\cdot 6=\\
&= 7\cdot 5^{2n}\cdot(19+6)+12\cdot 6^{n}\cdot 6=\\
&= 7\cdot 5^{2n}\cdot 19+7\cdot 5^{2n}\cdot 6+12\cdot 6^{n}\cdot 6=\\
&= 7\cdot 5^{2n}\cdot 19+6\cdot\left(7\cdot 5^{2n}+12\cdot 6^{n}\right)=\\
&= 7\cdot 5^{2n}\cdot 19+6\cdot 19A=19\left(7\cdot 5^{2n}+6A\right)=19B.
\end{aligned}$$

gdje je $B=7\cdot 5^{2n}+6A$.
Dakle, jednakost vrijedi i za $n=k+1$.
4) Jednakost vrijedi za svaki prirodan broj n.

Zadatak 5.9 *Dokazati da za svaki prirodan broj* n *vrijedi*

$$\sin x+\sin 3x+\sin 5x+\cdots+\sin(2n-1)x=\frac{\sin^2 nx}{\sin x}.$$

Rješenje:
1) Provjerimo da li jednakost vrijedi za $n=1$:
$\mathcal{L}\equiv\sin x$
$\mathcal{D}\equiv\frac{\sin^2 x}{\sin x}=\sin x$
Dakle, jednakost vrijedi za $n=1$.
2) Pretpostavimo da tvrdnja vrijedi za $n=k$, tj. neka vrijedi

$$\sin x+\sin 3x+\sin 5x+\cdots+\sin(2k-1)x=\frac{\sin^2 kx}{\sin x}.$$

3) Dokažimo da tvrdnja vrijedi i za $n=k+1$, tj. da vrijedi

$$\sin x+\sin 3x+\sin 5x+\cdots+\sin(2(k+1)-1)x=\frac{\sin^2(k+1)x}{\sin x}$$

Dokaz:

$$\sin x+\sin 3x+\sin 5x+\cdots+\sin(2k-1)x+\sin(2(k+1)-1)x=$$

$$\begin{aligned}
&= \frac{\sin^2 kx}{\sin x}+\sin(2k+1)x=\frac{\sin^2 kx+\sin x\cdot\sin(2k+1)x}{\sin x}=\\
&= \frac{\sin^2 kx+\frac{1}{2}\left[\cos 2kx-\cos(2kx+2x)\right]}{\sin x}=\\
&= \frac{\frac{2\sin^2 kx+\cos 2kx-\cos(2kx+2x)}{2}}{\sin x}=\frac{2\sin^2 kx+\cos 2kx-\cos 2(k+1)x}{2\sin x}=\\
&= \frac{2\sin^2 kx+\cos^2 kx-\sin^2 kx-\cos^2(k+1)x+\sin^2(k+1)x}{2\sin x}=\\
&= \frac{\sin^2 kx+\cos^2 kx-\cos^2(k+1)x+\sin^2(k+1)x}{2\sin x}=\\
&= \frac{1-\cos^2(k+1)x+\sin^2(k+1)x}{2\sin x}=\frac{\sin^2(k+1)x+\sin^2(k+1)x}{2\sin x}=\\
&= \frac{2\sin^2(k+1)x}{2\sin x}=\frac{\sin^2(k+1)x}{\sin x}.
\end{aligned}$$

Dakle, jednakost vrijedi i za $n=k+1$.
4) Jednakost vrijedi za svaki prirodan broj n.
Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n}\sin(2k-1)x=\frac{\sin^2 nx}{\sin x}.$$